Toroidal groups, generalized Jacobians and non-totally real number fields

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Abstract

A *toroidal group* is a complex Lie group on which every holomorphic function is constant.

In the first part of this talk, the relationship between toroidal groups and non-totally real number fields is introduced: under some necessary assumptions, any toroidal group $\mathcal{T}$ of complex dimension 2 and real rank 3 with extra multiplication is isogenous to the group $\mathbb{C}^2/\mu_\Phi(\mathcal{O}_K)$, where K is the non-totally real cubic number field $\mathrm{End}_0(\mathcal{T}) = \mathrm{End}(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and μ_Φ is a Minkowski embedding. This correspondence is extended to higher dimension, writing down the relations between the periods defining a toroidal group $\mathcal{T}$ (of complex dimension n and real rank $n+1$) and the minimal polynomial of a primitive element of K. Furthermore, for such a toroidal group $\mathcal{T}$, I explicitly show the analytic and rational representations of its ring of endomorphisms. Lastly, I show some results concerning the relationship between a toroidal groups of complex dimennsion 3 and real rank 5 and a quintic field with two pair of complex embeddings.

In the second part of this talk, I introduce the (birationally) isomorphism between a toroidal group and the generalized Jacobian of an elliptic (or hyperlliptic) curve. Finally, I give an explicit description of the m-torsion in the geometric correspondence of a toroidal group $\mathcal{T} = \mathbb{C}^2/\mu_\Phi(\mathcal{O}_K)$ with a generalized Jacobian $\mathfrak{J}$ of an elliptic curve.

Keywords: Toroidal groups; Jacobians; number fields.
MSC: 32M05, 22E10, 14H40, 57T15, 11G15.

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Biderivations of complete Lie algebras

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The main result of the paper [1] describes all complete Lie algebras with trivial center and internal derivations, i.e., a Lie algebra with a trivial center and with inner derivations. In particular, semi-simple Lie algebras fall into this category. The work also extends a well-known result on simple Lie algebras obtained by X. Tang in 2018 ([2]). Furthermore, the paper presents some results on symmetric and antisymmetric biderivations.

Proposition 1.1. *Let L be a complete Lie algebra over a field $\mathbb{F}$. B is a biderivation of L if and only if there exist two linear maps $\varphi, \psi \in \operatorname{End}(L)$ such that, for every $x, y \in L$,*

$$B(x, y) = [\varphi(x), y] = [x, \psi(y)] .$$

Theorem 1.2. *Let $L = L_1 \oplus \cdots \oplus L_t$ be a complex semisimple Lie algebra of dimension n, where L_i is a complex simple Lie algebra with $\dim_{\mathbb{C}} L_i = n_i$ for each $i = 1, \ldots, t$, and $n_1 + \cdots + n_t = n$. A bilinear map $B\colon L \times L \to L$ is a biderivation of L if and only if there exist $\lambda_1, \ldots, \lambda_t \in \mathbb{C}$ such that*

$$B(x_1 + \cdots + x_t, y_1 + \cdots + y_t) = \lambda_1 [x_1, y_1] + \cdots + \lambda_t [x_t, y_t] ,$$

with $x_i, y_i \in L_i$, $i = 1, \ldots, t$.

It is worth noting that a linear map $f : L \to L$ is called *commutative* if $[x, f(x)] = 0$ for every $x \in L$. If $\operatorname{char}(\mathbb{F}) \neq 2$, then a linear map $f\colon L \to L$ is commutative if and only if $[f(x), y] = [x, f(y)]$ for every $x, y \in L$. Similarly, f is called *antisymmetric* if and only if $[f(x), y] = -[x, f(y)]$ for all $x, y \in L$. To be more precise, the definition of commutative linear maps can be extended to a broader class of algebraic structures, such as rings.

Corollary 1.3. *Let L be a complete Lie algebra. $B\colon L \to L$ is a symmetric biderivation of L if and only if there exists a unique antisymmetric linear map $\varphi \in \operatorname{End}(L)$ such that $B(x, y) = [\varphi(x), y]$, for every $x, y \in L$.*

Corollary 1.4. *Let L be a complete Lie algebra. $B\colon L \to L$ is an antisymmetric biderivation of L if and only if there exists a unique commutative linear map $\varphi \in \operatorname{End}(L)$ such that $B(x, y) = [\varphi(x), y]$, for every $x, y \in L$.*

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Fusion Systems and their applications in block theoretic conjectures

Fusion systems arise from both finite groups and finite group blocks. A conjecture suggests the equivalence of these construction approaches. We explore the conjectures implications in Block Theory and discuss the broader applications of fusion systems, particularly in addressing key conjectures within this field.

Weak representability of actions: a study from groups to non-associative algebras

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It is well known that in the semi-abelian category **Grp** of groups, internal actions are represented by authomorphisms. This means that the category **Grp** is *action representable* and the representing object, which is called the *actor*, is the group of authomorphisms. Another example of action representable category is the variety **Lie** of Lie algebras over a fixed field $\mathbb{F}$, with the actor of a Lie algebra $\mathfrak{g}$ being the Lie algebra of derivations $\mathrm{Der}(\mathfrak{g})$. The notion of action representable category has proven to be quite restrictive: for instance, if a non-abelian variety $\mathcal{V}$ of non-associative algebras over an infinite field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$, is action representable, then $\mathcal{V} = \mathbf{Lie}$. More recently G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of categories, such as the variety **Assoc** of associative algebras and the variety **Leib** of Leibniz algebras.

In this talk we show that for an *algebraically coherent* and *operadic* variety $\mathcal{V}$ and an object X of $\mathcal{V}$, it is always possible to construct a *partial algebra* $\mathcal{E}(X)$, called *external weak actor* of X, and a natural monomorphism of functors

$$\tau\colon \mathrm{Act}(-,X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-),\mathcal{E}(X)),$$

where **PAlg** is the category of partial algebras over $\mathbb{F}$ and $U\colon \mathcal{V} \to \mathbf{PAlg}$ denotes the forgetful functor. The pair $(\mathcal{E}(X),\tau)$ is called *external weak representation* of the functor $\mathrm{Act}(-,X)$. Moreover, for any other object B of $\mathcal{V}$, we provide a complete description of the morphisms $(B \to \mathcal{E}(X)) \in \mathrm{Im}(\tau_B)$, i.e. of the homomorphisms of partial algebras which identify the actions of B on X in $\mathcal{V}$.

Eventually, we give an application of this construction in the context of varieties of *unital* algebras: we prove that, if $\mathcal{V} = \mathbf{Alt}$ is the variety of *alternative* algebras and X is a unital alternative algebra, then $\mathcal{E}(X) \cong X$ is the actor of X. In other words, unital alternative algebras, such as the algebra $\mathbb{O}$ of *octonions*, have representable actions.

This is joint work with Alan S. Cigoli (*Università degli Studi di Torino*, Italy), Xabier García Martínez (*Universidade de Vigo*, Spain), Giuseppe Metere (*Università degli Studi di Palermo*, Italy), Tim Van der Linden and Corentin Vienne (*Université catholique de Louvain*, Belgium).

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An extension formula for right Bol loops arising from Bol reflections

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Joint work with Gabor P. Nagy

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We study a new extension formula for right Bol loops. We prove the necessary and sufficient conditions for the extension to be right Bol. We describe the most important invariants: right multiplication group, nuclei, center. We show that the core is an involutory quandle which is the disjoint union of two isomorphic involutory quandles. We also derive further results on the structure group of the core of the extension.

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On a Group Action of Subgroups of PGL_2 on Monic Polynomials

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Abstract

Let K be a field, $K(x)$ the rational function field over K and $\mathrm{PGL}_2(K)$ the projective general linear group over K. We write $[A]$ for the coset of $A \in \mathrm{GL}_2(K)$ in $\mathrm{PGL}_2(K)$. For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we define $[A] \circ x = \frac{ax+b}{cx+d}$ and set

$$f^{[A]}(x) := \lambda_{[A],f}(cx+d)^{\deg(f)} f\left([A] \circ x\right),$$

where $\lambda_{[A],f} \in K^*$ makes the output-polynomial monic. It can be shown that this induces a right group action of $G \leq \mathrm{PGL}_2(K)$ on a subset of monic polynomials over K.

We present some recent developments about polynomials that are invariant under this action and show how they are related to *rational transformations* of polynomials, a concept arising in the context of the construction of irreducible polynomials over finite fields.

Keywords: Group Action, Factorization of Polynomials, Galois Theory

Classic Jacobian of elliptic curves over local fields as generalized Jacobian

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In this talk, I will describe an algebro-geometric interpretation of semi-periodic elliptic functions, in the frame of the short exact sequence defined by the reduction modulo p of an elliptic curve over the field of p-adics, whose kernel is known to be the additive group of p-adic integers.